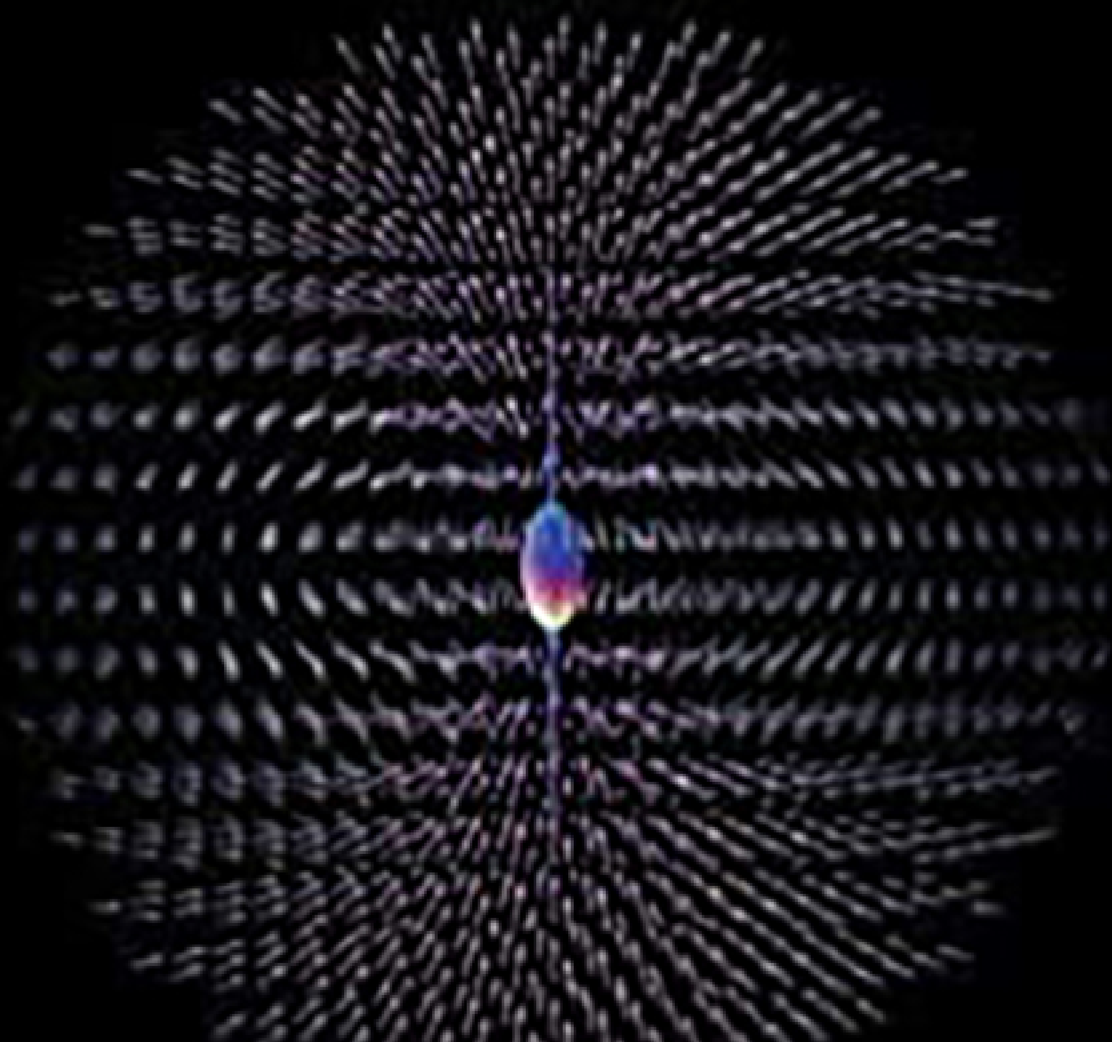
ENGINEERING ELECTROMAGNETICS

ninth edition



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McGraw Hill Education

Engineering Electromagnetics

NINTH EDITION

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ABOUT THE AUTHORS

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A native of Los Angeles, California, **John A. Buck** received his B.S. in Engineering from UCLA in 1975, and the M.S. and Ph.D. degrees in Electrical Engineering from the University of California at Berkeley in 1977 and 1982. In 1982, he joined the faculty of the School of Electrical and Computer Engineering at Georgia Tech, and is now Professor Emeritus. His research areas and publications have centered within the fields of ultrafast switching, nonlinear optics, and optical fiber communications. He is the author of the graduate text *Fundamentals of Optical Fibers* (Wiley Interscience), which is in its second edition. Dr. Buck is the recipient of four institute teaching awards and the IEEE Third Millenium Medal.

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PREFACE

The printing of this book occurs one year short of 60 years since its first edition, which was at that time under the sole authorship of William H. Hayt, Jr. In a sense, I grew up with the book, having used the second edition in a basic electromagnetics course as a college junior. The reputation of the subject matter precedes itself. The prospect of taking the first course in electromagnetics was then, as now, a matter of dread to many if not most. One professor of mine at Berkeley put it succinctly through the rather negative observation that electromagetics is "a test of your ability to bend your mind". But on entering the course and first opening the book, I was surprised and relieved to find the friendly writing style and the measured approach to the subject. This for me made it a very readable book, out of which I was able to learn with little help from my instructor. I referred to the book often while in graduate school, taught from the fourth and fifth editions as a faculty member, and then became coauthor for the sixth edition on the retirement (and subsequent untimely death) of Bill Hayt. To this day, the memories of my time as a beginner are vivid, and in preparing the sixth and subsequent editions, I have tried to maintain the accessible style that I found so encouraging and useful then.

Over the 60-year span, the subject matter has not changed, but emphases have. In universities, the trend continues toward reducing electrical engineering core course allocations to electromagnetics. This is a matter of economy, rather than any belief in diminished relevance. Quite the contrary: A knowledge of electromagnetic field theory is in the present day more important than ever for the electrical engineer. Examples that demonstrate this include the continuing expansion of high-speed wireless and optical fiber communication. Additionally, the need continues for ever-smaller and denser microcircuitry, in which a command of field theory is essential for successful designs. The more traditional applications of electrical power generation and distribution remain as important as ever.

I have made efforts to further improve the presentation in this new edition. Most changes occur in the earlier chapters, in which much of the wording has been shortened, and several explanations were improved. Additional introductory material has been added in several places to provide perspective. In addition, all chapters are now subsectioned, to improve the organization and to make topics easier to locate.

Some 100 new end-of-chapter problems have been added throughout, all of which replaced older problems that I considered well-worn. For some of these, I chose particularly good "classic" problems from the earliest editions. I have retained the previous system in which the approximate level of difficulty is indicated beside each problem on a three-level scale. The lowest level is considered a fairly straightforward problem, requiring little work assuming the material is understood; a level 2 problem is conceptually more difficult, and/or may require more work to solve; a level 3 problem is considered either difficult conceptually, or may require extra effort (including possibly the help of a computer) to solve.

As in the previous edition, the transmission lines chapter (10) is stand-alone, and can be read or covered in any part of a course, including the beginning. In it, transmission lines are treated entirely within the context of circuit theory; wave phenomena are introduced and used exclusively in the form of voltages and currents. Inductance and capacitance concepts are treated as known parameters, and so there is no reliance on any other chapter. Field concepts and parameter computation in transmission lines appear in the early part of the waveguides chapter (13), where they play additional roles of helping to introduce waveguiding concepts. The chapters on electromagnetic waves, 11 and 12, retain their independence of transmission line theory in that one can progress from Chapter 9 directly to Chapter 11. By doing this, wave phenomena are introduced from first principles but within the context of the uniform plane wave. Chapter 11 refers to Chapter 10 in places where the latter may give additional perspective, along with a little more detail. Nevertheless, all necessary material to learn plane waves without previously studying transmission line waves is found in Chapter 11, should the student or instructor wish to proceed in that order.

The antennas chapter covers radiation concepts, building on the retarded potential discussion in Chapter 9. The discussion focuses on the dipole antenna, individually and in simple arrays. The last section covers elementary transmit-receive systems, again using the dipole as a vehicle.

The book is designed optimally for a two-semester course. As is evident, statics concepts are emphasized and occur first in the presentation, but again Chapter 10 (transmission lines) can be read first. In a single course that emphasizes dynamics, the transmission lines chapter can be covered initially as mentioned or at any point in the course. One way to cover the statics material more rapidly is by deemphasizing materials properties (assuming these are covered in other courses) and some of the advanced topics. This involves omitting Chapter 1 (assigned to be read as a review), and omitting Sections 2.5, 2.6, 4.7, 4.8, 5.5–5.7, 6.3, 6.4, 6.7, 7.6, 7.7, 8.5, 8.6, 8.8, 8.9, and 9.5.

A supplement to this edition is web-based material consisting of articles on special topics in addition to animated demonstrations and interactive programs developed by Natalya Nikolova of McMaster University and Vikram Jandhyala of the University of Washington. Their excellent contributions are geared to the text, and icons appear in the margins whenever an exercise that pertains to the narrative exists. In addition, quizzes are provided to aid in further study.

The theme of the text is the same as it has been since the first edition of 1958. An inductive approach is used that is consistent with the historical development. In it, the experimental laws are presented as individual concepts that are later unified in Maxwell's equations. After the first chapter on vector analysis, additional mathematical tools are introduced in the text on an as-needed basis. Throughout every edition, as well as this one, the primary goal has been to enable students to learn independently. Numerous examples, drill problems (usually having multiple parts), end-of-chapter problems, and material on the web site, are provided to facilitate this.

Answers to the drill problems are given below each problem. Answers to odd-numbered end-of-chapter problems are found in Appendix F. A solutions manual and a set of PowerPoint slides, containing pertinent figures and equations, are available to instructors. These, along with all other material mentioned previously, can be accessed on the website:

www.mhhe.com/haytbuck

I would like to acknowledge the valuable input of several people who helped to make this a better edition. They include:

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I also acknowledge the feedback and many comments from students, too numerous to name, including several who have contacted me from afar. I continue to be open and grateful for this feedback and can be reached at john.buck@ece.gatech.edu. Many suggestions were made that I considered constructive and actionable. I regret that not all could be incorporated because of time restrictions. Creating this book was a team effort, involving several outstanding people at McGraw-Hill. These include my editors, Raghu Srinivasan and Tomm Scaife, whose vision and encouragement

were invaluable. Jenilynn McAtee and Lora Neyens deftly coordinated the production phase with excellent ideas and enthusiasm, and Tina Bower, who was my guide and conscience from the beginning, providing valuable insights, and jarring me into action when necessary. I am, as usual in these projects, grateful to a patient and supportive family.

John A. Buck
Marietta, Georgia
May, 2017

CHAPTER 1

Vector Analysis

Vector analysis is a subject that is better taught by mathematicians than by engineers. Most junior and senior engineering students have not had the time (or the inclination) to take a course in vector analysis, although it is likely that vector concepts and operations were introduced in the calculus courses. These are covered in this chapter, and the time devoted to them now should depend on past exposure.

The viewpoint here is that of the engineer or physicist and not that of the mathematician. Proofs are indicated rather than rigorously expounded, and physical interpretation is stressed. It is easier for engineers to take a more rigorous course in the mathematics department after they have been presented with a few physical pictures and applications.

Vector analysis is a mathematical shorthand. It has some new symbols and some new rules, and it demands concentration and practice. The drill problems, first found at the end of Section 1.4, should be considered part of the text and should all be worked. They should not prove to be difficult if the material in the accompanying section of the text has been thoroughly understood. ■

1.1 SCALARS AND VECTORS

The term *scalar* refers to a quantity whose value may be represented by a single (positive or negative) real number. The x, y, and z we use in basic algebra are scalars, as are the quantities they represent. If we speak of a body falling a distance L in a time t, or the temperature T at any point whose coordinates are x, y, and z, then L, t, T, x, y, and z are all scalars. Other scalar quantities are mass, density, pressure (but not force), volume, volume resistivity, and voltage.

A *vector* quantity has both a magnitude[1] and a direction in space. We are concerned with two- and three-dimensional spaces only, but vectors may be defined in

[1] We adopt the convention that magnitude infers absolute value; the magnitude of any quantity is therefore always positive.

n-dimensional space in more advanced applications. Force, velocity, acceleration, and a straight line from the positive to the negative terminal of a storage battery are examples of vectors. Each quantity is characterized by both a magnitude and a direction.

Our work will mainly concern scalar and vector *fields*. A field (scalar or vector) may be defined mathematically as some function that connects an arbitrary origin to a general point in space. We usually associate some physical effect with a field, such as the force on a compass needle in the earth's magnetic field, or the movement of smoke particles in the field defined by the vector velocity of air in some region of space. Note that the field concept invariably is related to a region. Some quantity is defined at every point in a region. Both *scalar fields* and *vector fields* exist. The temperature and the density at any point in the earth are examples of scalar fields. The gravitational and magnetic fields of the earth, the voltage gradient in a cable, and the temperature gradient in a soldering-iron tip are examples of vector fields. The value of a field varies in general with both position and time.

In this book, as in most others using vector notation, vectors will be indicated by boldface type, for example, **A**. Scalars are printed in italic type, for example, *A*. When writing longhand, it is customary to draw a line or an arrow over a vector quantity to show its vector character. (CAUTION: This is the first pitfall. Sloppy notation, such as the omission of the line or arrow symbol for a vector, is the major cause of errors in vector analysis.)

1.2 VECTOR ALGEBRA

In this section, the rules of vector arithmetic, vector algebra, and (later) vector calculus are defined. Some of the rules will be similar to those of scalar algebra, some will differ slightly, and some will be entirely new.

1.2.1 Addition and Subtraction

The addition of vectors follows the parallelogram law. Figure 1.1 shows the sum of two vectors, **A** and **B**. It is easily seen that $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$, or that vector addition obeys the commutative law. Vector addition also obeys the associative law,

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$$

Note that when a vector is drawn as an arrow of finite length, its location is defined to be at the tail end of the arrow.

Coplanar vectors are vectors lying in a common plane, such as those shown in Figure 1.1. Both lie in the plane of the paper and may be added by expressing each vector in terms of "horizontal" and "vertical" components and then adding the corresponding components.

Vectors in three dimensions may likewise be added by expressing the vectors in terms of three components and adding the corresponding components. Examples of this process of addition will be given after vector components are discussed in Section 1.4.

Figure 1.1 Two vectors may be added graphically either by drawing both vectors from a common origin and completing the parallelogram or by beginning the second vector from the head of the first and completing the triangle; either method is easily extended to three or more vectors.

The rule for the subtraction of vectors follows easily from that for addition, for we may always express $\mathbf{A} - \mathbf{B}$ as $\mathbf{A} + (-\mathbf{B})$; the sign, or direction, of the second vector is reversed, and this vector is then added to the first by the rule for vector addition.

1.2.2 Multiplication and Division

Vectors may be multiplied by scalars. The magnitude of the vector changes, but its direction does not when the scalar is positive, although it reverses direction when multiplied by a negative scalar. Multiplication of a vector by a scalar also obeys the associative and distributive laws of algebra, leading to

$$(r + s)(\mathbf{A} + \mathbf{B}) = r(\mathbf{A} + \mathbf{B}) + s(\mathbf{A} + \mathbf{B}) = r\mathbf{A} + r\mathbf{B} + s\mathbf{A} + s\mathbf{B}$$

Division of a vector by a scalar is merely multiplication by the reciprocal of that scalar. The multiplication of a vector by a vector is discussed in Sections 1.6 and 1.7. Two vectors are said to be equal if their difference is zero, or $\mathbf{A} = \mathbf{B}$ if $\mathbf{A} - \mathbf{B} = 0$.

In our use of vector fields we always add and subtract vectors that are defined at the same point. For example, the *total* magnetic field about a small horseshoe magnet will be shown to be the sum of the fields produced by the earth and the permanent magnet; the total field at any point is the sum of the individual fields *at that point*.

1.3 THE RECTANGULAR COORDINATE SYSTEM

To describe a vector accurately, some specific lengths, directions, angles, projections, or components must be given. There are three simple coordinate systems by which this is done, and about eight or ten other systems that are useful in very special cases. We are going to use only the three simple systems, the simplest of which is the *rectangular*, or *rectangular cartesian*, coordinate system.

1.3.1 Right-Handed Coordinate Systems

In the rectangular coordinate system we set up three coordinate axes mutually at right angles to each other and call them the x, y, and z axes. It is customary to choose a *right-handed* coordinate system, in which a rotation (through the smaller angle) of the x axis into the y axis would cause a right-handed screw to progress in the direction of the z axis. If the right hand is used, then the thumb, forefinger, and

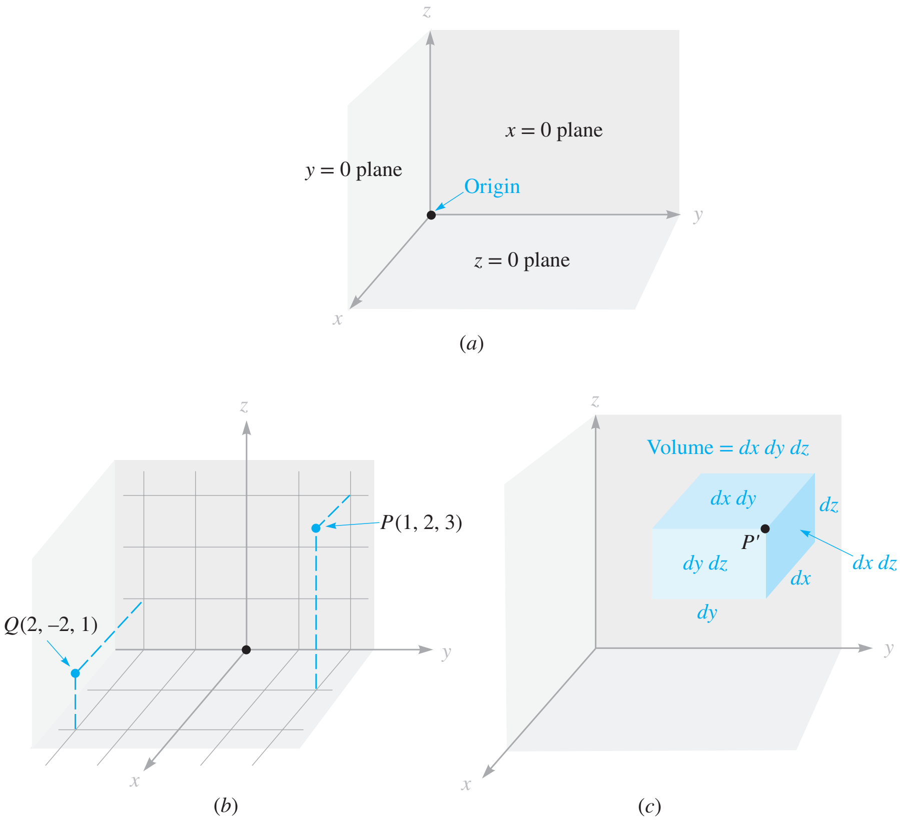

Figure 1.2 (*a*) A right-handed rectangular coordinate system. If the curved fingers of the right hand indicate the direction through which the *x* axis is turned into coincidence with the *y* axis, the thumb shows the direction of the *z* axis. (*b*) The location of points *P*(1, 2, 3) and *Q*(2, −2, 1). (*c*) The differential volume element in rectangular coordinates; *dx*, *dy*, and *dz* are, in general, independent differentials.

middle finger may be identified, respectively, as the x, y, and z axes. Figure 1.2*a* shows a right-handed rectangular coordinate system. A point is located by giving its x, y, and z coordinates. These are, respectively, the distances from the origin to the intersection of perpendicular lines dropped from the point to the x, y, and z axes.

1.3.2 Point Locations as Intersections of Planes

An alternative method of interpreting coordinate values, which *must* be used in all other coordinate systems, is to consider a point as being at the common intersection of three surfaces. In rectangular coordinates, these are the planes $x =$ constant, $y =$ constant, and $z =$ constant, where the constants are the coordinate values of the point.

Figure 1.2*b* shows points P and Q whose coordinates are (1, 2, 3) and (2, −2, 1), respectively. Point P is therefore located at the common point of intersection of the planes $x = 1$, $y = 2$, and $z = 3$, whereas point Q is located at the intersection of the planes $x = 2$, $y = -2$, and $z = 1$.

In other coordinate systems, as discussed in Sections 1.8 and 1.9, we expect points to be located at the common intersection of three surfaces, not necessarily planes, but still mutually perpendicular at the point of intersection.

If we visualize three planes intersecting at the general point P, whose coordinates are x, y, and z, we may increase each coordinate value by a differential amount and obtain three slightly displaced planes intersecting at point P', whose coordinates are $x + dx$, $y + dy$, and $z + dz$. The six planes define a rectangular parallelepiped whose volume is $dv = dxdydz$; the surfaces have differential areas dS of $dxdy$, $dydz$, and $dzdx$. Finally, the distance dL from P to P' is the diagonal of the parallelepiped and has a length of $\sqrt{(dx)^2 + (dy)^2 + (dz)^2}$. The volume element is shown in Figure 1.2*c*; point P' is indicated, but point P is located at the only invisible corner.

All this is familiar from trigonometry or solid geometry and as yet involves only scalar quantities. We will describe vectors in terms of a coordinate system in the next section.

1.4 VECTOR COMPONENTS AND UNIT VECTORS

To describe a vector in the rectangular coordinate system, first consider a vector $\mathbf{r}$ extending outward from the origin. A logical way to identify this vector is by giving the three *component vectors*, lying along the three coordinate axes, whose vector sum must be the given vector. If the component vectors of the vector $\mathbf{r}$ are $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$, then $\mathbf{r} = \mathbf{x} + \mathbf{y} + \mathbf{z}$. The component vectors are shown in Figure 1.3*a*. Instead of one vector, we now have three, but this is a step forward because the three vectors are of a very simple nature; each is always directed along one of the coordinate axes.

The component vectors in Figure 1.3 have magnitudes that depend on the given vector (such as $\mathbf{r}$), but they each have a known and constant direction. This suggests the use of *unit vectors* having unit magnitude by definition; these are parallel to the coordinate axes and they point in the direction of increasing coordinate values. We reserve the symbol $\mathbf{a}$ for a unit vector and identify its direction by an appropriate subscript. Thus $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ are the unit vectors in the rectangular coordinate system.[2] They are directed along the x, y, and z axes, respectively, as shown in Figure 1.3*b*.

If the component vector $\mathbf{y}$ happens to be two units in magnitude and directed toward increasing values of y, we then write $\mathbf{y} = 2\mathbf{a}_y$. A vector $\mathbf{r}_P$ pointing from the origin to point $P(1, 2, 3)$ is written $\mathbf{r}_P = \mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z$. The vector from P to Q is obtained by applying the rule of vector addition. This rule shows that the vector from the origin to P plus the vector from P to Q is equal to the vector from the origin to Q. The desired vector from $P(1, 2, 3)$ to $Q(2, -2, 1)$ is therefore

$$\begin{aligned}\mathbf{R}_{PQ} = \mathbf{r}_Q - \mathbf{r}_P &= (2-1)\mathbf{a}_x + (-2-2)\mathbf{a}_y + (1-3)\mathbf{a}_z \\ &= \mathbf{a}_x - 4\mathbf{a}_y - 2\mathbf{a}_z\end{aligned}$$

The vectors $\mathbf{r}_P$, $\mathbf{r}_Q$, and $\mathbf{R}_{PQ}$ are shown in Figure 1.3*c*.

[2] The symbols **i**, **j**, and **k** are also commonly used for the unit vectors in rectangular coordinates.

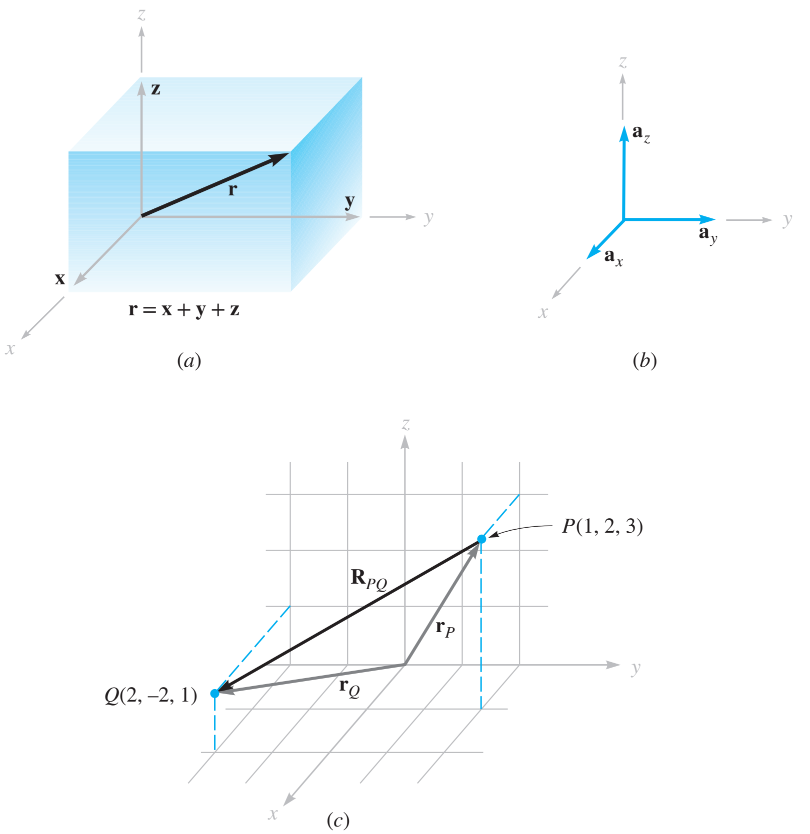

Figure 1.3 (*a*) The component vectors **x**, **y**, and **z** of vector **r**. (*b*) The unit vectors of the rectangular coordinate system have unit magnitude and are directed toward increasing values of their respective variables. (*c*) The vector $\mathbf{R}_{PQ}$ is equal to the vector difference $\mathbf{r}_Q - \mathbf{r}_P$.

The last vector does not extend outward from the origin, as did the vector **r** we initially considered. However, we have already learned that vectors having the same magnitude and pointing in the same direction are equal, so we see that to help our visualization processes we are at liberty to slide any vector over to the origin before determining its component vectors. Parallelism must, of course, be maintained during the sliding process.

In discussing a force vector **F**, or any vector other than a displacement-type vector such as **r**, the problem arises of providing suitable letters for the three component vectors. It would not do to call them **x**, **y**, and **z**, for these are displacements, or directed distances, and are measured in meters (abbreviated m) or some other unit of length. The problem is most often avoided by using *component scalars*, simply called *components*, F_x,F_y, and F_z. The components are the signed magnitudes of the component vectors. We may then write $\mathbf{F} = F_x\mathbf{a}_x + F_y\mathbf{a}_y + F_z\mathbf{a}_z$. The component vectors are $F_x\mathbf{a}_x$, $F_y\mathbf{a}_y$, and $F_z\mathbf{a}_z$.

Any vector **B** then may be described by $\mathbf{B} = B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z$. The magnitude of **B** written $|\mathbf{B}|$ or simply B, is given by

$$|\mathbf{B}| = \sqrt{B_x^2 + B_y^2 + B_z^2} \tag{1}$$

Each of the three coordinate systems we discuss will have its three fundamental and mutually perpendicular unit vectors that are used to resolve any vector into its component vectors. Unit vectors are not limited to this application. It is helpful to write a unit vector having a specified direction. This is easily done, for a unit vector in a given direction is merely a vector in that direction divided by its magnitude. A unit vector in the **r** direction is $\mathbf{r}/\sqrt{x^2 + y^2 + z^2}$, and a unit vector in the direction of the vector **B** is

$$\mathbf{a}_B = \frac{\mathbf{B}}{\sqrt{B_x^2 + B_y^2 + B_z^2}} = \frac{\mathbf{B}}{|\mathbf{B}|} \tag{2}$$

EXAMPLE 1.1

Specify the unit vector extending from the origin toward the point $G(2, -2, -1)$.

Solution. We first construct the vector extending from the origin to point G,

$$\mathbf{G} = 2\mathbf{a}_x - 2\mathbf{a}_y - \mathbf{a}_z$$

We continue by finding the magnitude of **G**,

$$|\mathbf{G}| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = 3$$

and finally expressing the desired unit vector as the quotient,

$$\mathbf{a}_G = \frac{\mathbf{G}}{|\mathbf{G}|} = \frac{2}{3}\mathbf{a}_x - \frac{2}{3}\mathbf{a}_y - \frac{1}{3}\mathbf{a}_z = 0.667\,\mathbf{a}_x - 0.667\,\mathbf{a}_y - 0.333\,\mathbf{a}_z$$

A special symbol is desirable for a unit vector so that its character is immediately apparent. Symbols that have been used are $\mathbf{u}_B$, $\mathbf{a}_B$, $\mathbf{1}_B$, or even **b**. We will consistently use the lowercase **a** with an appropriate subscript.

[NOTE: Throughout the text, drill problems appear following sections in which a new principle is introduced in order to allow students to test their understanding of the basic fact itself. The problems are useful in gaining familiarity with new terms and ideas and should all be worked. More general problems appear at the ends of the chapters. The answers to the drill problems are given in the same order as the parts of the problem.]

D1.1. Given points $M(-1, 2, 1)$, $N(3, -3, 0)$, and $P(-2, -3, -4)$, find: (*a*) $\mathbf{R}_{MN}$; (*b*) $\mathbf{R}_{MN} + \mathbf{R}_{MP}$; (*c*) $|\mathbf{r}_M|$; (*d*) $\mathbf{a}_{MP}$; (*e*) $|2\mathbf{r}_P - 3\mathbf{r}_N|$.

Ans. (*a*) $4\mathbf{a}_x - 5\mathbf{a}_y - \mathbf{a}_z$; (*b*) $3\mathbf{a}_x - 10\mathbf{a}_y - 6\mathbf{a}_z$; (*c*) 2.45; (*d*) $-0.14\mathbf{a}_x - 0.7\mathbf{a}_y - 0.7\mathbf{a}_z$; (*e*) 15.56

1.5 THE VECTOR FIELD

We have defined a vector field as a vector function of a position vector. In general, the magnitude and direction of the function will change as we move throughout the region, and the value of the vector function must be determined using the coordinate values of the point in question. In the rectangular coordinate system, the vector will be a function of the variables x, y, and z.

Again, representing the position vector as $\mathbf{r}$, a vector field $\mathbf{G}$ can be expressed in functional notation as $\mathbf{G}(\mathbf{r})$; a scalar field T is written as $T(\mathbf{r})$.

If we inspect the velocity of the water in the ocean in some region near the surface where tides and currents are important, we might decide to represent it by a velocity vector that is in any direction, even up or down. If the z axis is taken as upward, the x axis in a northerly direction, the y axis to the west, and the origin at the surface, we have a right-handed coordinate system and may write the velocity vector as $\mathbf{v} = v_x\mathbf{a}_x + v_y\mathbf{a}_y + v_z\mathbf{a}_z$, or $\mathbf{v}(\mathbf{r}) = v_x(\mathbf{r})\mathbf{a}_x + v_y(\mathbf{r})\mathbf{a}_y + v_z(\mathbf{r})\mathbf{a}_z$; each of the components v_x, v_y, and v_z may be a function of the three variables x, y, and z. If we are in some portion of the Gulf Stream where the water is moving only to the north, then v_y and v_z are zero. Further simplifying assumptions might be made if the velocity falls off with depth and changes very slowly as we move north, south, east, or west. A suitable expression could be $\mathbf{v} = 2e^{z/100}\mathbf{a}_x$. We have a velocity of 2 m/s (meters per second) at the surface and a velocity of 0.368×2, or 0.736 m/s, at a depth of 100 m ($z = -100$). The velocity continues to decrease with depth while maintaining a constant direction.

D1.2. A vector field $\mathbf{S}$ is expressed in rectangular coordinates as $\mathbf{S} = \{125/[(x-1)^2 + (y-2)^2 + (z+1)^2]\}\{(x-1)\mathbf{a}_x + (y-2)\mathbf{a}_y + (z+1)\mathbf{a}_z\}$. ($a$) Evaluate $\mathbf{S}$ at $P(2, 4, 3)$. (b) Determine a unit vector that gives the direction of $\mathbf{S}$ at P. (c) Specify the surface $f(x, y, z)$ on which $|\mathbf{S}| = 1$.

Ans. (a) $5.95\mathbf{a}_x + 11.90\mathbf{a}_y + 23.8\mathbf{a}_z$; ($b$) $0.218\mathbf{a}_x + 0.436\mathbf{a}_y + 0.873\mathbf{a}_z$; ($c$) $\sqrt{(x-1)^2 + (y-2)^2 + (z+1)^2} = 125$

1.6 THE DOT PRODUCT

The *dot product* (or scalar product) is used to multiply a given vector field by the component of another field that is *parallel* to the first. This gives the same result when the roles of the fields are reversed. In that sense, the dot product is a projection operation, which can be used to obtain the magnitude of a given field in a specific direction in space.

1.6.1 Geometric Definition

Given two vectors $\mathbf{A}$ and $\mathbf{B}$, the dot product is geometrically defined as the product of the magnitude of $\mathbf{A}$, the magnitude of $\mathbf{B}$, and the cosine of the smaller angle between them, thus projecting one field onto the other:

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta_{AB} \tag{3}$$

The dot appears between the two vectors and should be made heavy for emphasis. The dot, or scalar, product is a scalar, as one of the names implies, and it obeys the commutative law,

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \tag{4}$$

for the sign of the angle does not affect the cosine term. The expression $\mathbf{A} \cdot \mathbf{B}$ is read "**A** dot **B**."

A common application of the dot product is in mechanics, where a constant force **F** applied over a straight displacement **L** does an amount of work $FL \cos \theta$, which is more easily written $\mathbf{F} \cdot \mathbf{L}$. If the force varies along the path, integration is necessary to find the total work (as is taken up in Chapter 4), and the result becomes

$$\text{Work} = \int \mathbf{F} \cdot d\mathbf{L}$$

Another example occurs in magnetic fields. The total flux Φ crossing a surface of area S is given by BS if the magnetic flux density B is perpendicular to the surface and uniform over it. We define a *vector surface* **S** as having area for its magnitude and having a direction *normal* to the surface (avoiding for the moment the problem of which of the two possible normals to take). The flux crossing the surface is then $\mathbf{B} \cdot \mathbf{S}$. This expression is valid for any direction of the uniform magnetic flux density. If the flux density is not constant over the surface, the total flux is $\Phi = \int \mathbf{B} \cdot d\mathbf{S}$. Integrals of this general form appear in Chapter 3 in the context of electric flux density.

1.6.2 Operational Definition

Finding the angle between two vectors in three-dimensional space is often a job we would prefer to avoid, and for that reason the definition of the dot product is usually not used in its basic form. A more helpful result is obtained by considering two vectors whose rectangular components are given, such as $\mathbf{A} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$ and $\mathbf{B} = B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z$. The dot product also obeys the distributive law, and, therefore, $\mathbf{A} \cdot \mathbf{B}$ yields the sum of nine scalar terms, each involving the dot product of two unit vectors. Because the angle between two different unit vectors of the rectangular coordinate system is 90°, we then have

$$\mathbf{a}_x \cdot \mathbf{a}_y = \mathbf{a}_y \cdot \mathbf{a}_x = \mathbf{a}_x \cdot \mathbf{a}_z = \mathbf{a}_z \cdot \mathbf{a}_x = \mathbf{a}_y \cdot \mathbf{a}_z = \mathbf{a}_z \cdot \mathbf{a}_y = 0$$

The remaining three terms involve the dot product of a unit vector with itself, which is unity, giving finally the operational definition:

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z \tag{5}$$

which is an expression involving no angles.